EULER CHARACTERISTIC OF COHERENT SHEAVES ON SIMPLICIAL TORICS VIA THE STANLEY-REISNER RING

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Abstract. We combine work of Cox on the total coordinate ring of a toric variety and results of Eisenbud-Mustaţă-Stillman and Mustaţă on cohomology of toric and monomial ideals to obtain a formula for computing $\chi(\mathcal{O}_X(D))$ for a Weil divisor D on a complete simplicial toric variety X_Σ. The main point is to use Alexander duality to pass from the toric irrelevant ideal, which appears in the computation of $\chi(\mathcal{O}_X(D))$, to the Stanley-Reisner ideal of Σ, which is used in defining the Chow ring of X_Σ.

1. Introduction

For a divisor D on a smooth complete variety X, the Hirzebruch-Riemann-Roch theorem describes the Euler characteristic of $\mathcal{O}_X(D)$ in terms of intersection theory:

$$\chi(\mathcal{O}_X(D)) = \int ch(D) \cdot Td(X).$$

The divisor D corresponds to a class $[D]$ in the Chow ring of X, and $ch(D)$ consists of the first $n = dim(X)$ terms of the formal Taylor expansion of e^D. The Todd class of D is defined similarly, but using the Taylor expansion for $\frac{D}{1-e^{-D}}$. To define the Todd class of X, filter the tangent bundle $\mathcal{T}_X$ by line bundles $\mathcal{O}(D_i)$. Then one shows that $Td(X) = \prod_{i=1}^n Td(D_i)$ is independent of the filtration.

Let $\Sigma \subseteq \mathbb{R}^n$ be a complete simplicial rational polyhedral fan with $d = |\Sigma(1)|$ rays, X_Σ the associated toric variety, and $D \in \mathrm{Cl}(X_\Sigma)$ a Weil divisor on X_Σ. We combine Alexander duality and the Cox ring with results of Mustaţă [16] on monomial ideals to obtain a formula for the Euler characteristic of the associated rank one reflexive sheaf $\mathcal{O}_{X_\Sigma}(D)$. Put $Z = \{0,1\}^d$ and $\mathbf{1} = \{1\}^d$. Then for $l \gg 0$,

$$\chi(\mathcal{O}_X(D)) = \sum_{\mathbf{m} \in Z \setminus \mathbf{0}} (-1)^{|\mathbf{m}|-d+n} \dim_{\mathbb{K}}(S/I_\Sigma)_{\mathbf{1}-\mathbf{m}} \cdot \dim_{\mathbb{K}} S_{l\cdot\phi(\mathbf{m})+D}. \tag{1}$$

Here I_Σ denotes the Stanley-Reisner ideal, and $\mathbb{Z}^d \xrightarrow{\phi} \mathrm{Cl}(X_\Sigma)$ is the standard surjection of $\mathbb{Z}^d$ onto the class group. The Cox ring S is a polynomial ring, graded by $\mathrm{Cl}(X_\Sigma)$; on S/I_Σ we use the $\mathbb{Z}^d$ grading. We recall the definitions of these objects in §2. Any coherent sheaf on a nondegenerate toric variety corresponds to a finitely generated $\mathrm{Cl}(X_\Sigma)$-graded S–module (see [3] for the simplicial case, and [17] for the general case), so such a sheaf has a resolution by rank one reflexive sheaves, and Equation 1 yields a formula for $\chi(\mathcal{F})$ for any coherent sheaf $\mathcal{F}$. Bounds on l are determined by Eisenbud-Mustaţă-Stillman in [6], and are discussed in §2.

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Connections to physics and some history. The methods which are used to prove Equation 1 have applications to computations arising in mathematical physics: in a recent preprint [1], Blumenhagen, Jurke, Rahn and Roschy conjectured an algorithm for computing the cohomology of line bundles on a toric variety. Their motivation was to compute massless modes in Type IIB/F and heterotic compactifications, on a complete intersection in a toric variety. A strong form of the algorithm is established by Maclagan and Smith in Corollary 3.4 of [14]; later proofs appear in Jow [9] and Rahn-Roschy [20]. In all these papers Alexander duality and results of [6] play a key role, as they do in the proof of Equation 1. The original motivation for this work was to find a toric proof for the Hirzebruch-Riemann-Roch theorem.

The first toric interpretation of Hirzebruch-Riemann-Roch is due to Khovanskii [11]. In [12], [13], Pukhlikov-Khovanskii study additive measures on virtual polyhedra, and obtain a Riemann-Roch formula for integrating sums of quasipolynomials on virtual polytopes. Pommersheim [18] and Pommersheim and Thomas [19] obtain results on Todd classes of simplicial torics, and in [2], Brion-Vergne prove an equivariant Riemann-Roch for simplicial torics.

The results of Eisenbud-Mustaţă-Stillman in [6] show that in the toric setting, $\chi(\mathcal{O}_X(D))$ may be calculated via certain *Ext* modules over the Cox ring of X. On the other hand, evaluating the expression $\int ch(D) \cdot Td(X)$ involves a computation in the Chow ring of X, and the Cox and Chow rings of a simplicial toric variety are connected by Alexander duality.

The paper is structured as follows: in §2 we recall the results of [6] and the computation of cohomology via the Cox ring. In §3 we introduce the Chow ring, recall that the Stanley-Reisner ideal of Σ is the Alexander dual of the toric irrelevant ideal of Σ, and use results of Mustaţă and Stanley to connect the parts. Equation 1 is proved in §4, and illustrated on the Hirzebruch surface $\mathcal{H}_2$.

Toric facts. Let $N \simeq \mathbb{Z}^n$ be a lattice, with dual lattice M, and let $\Sigma \subseteq N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ be a complete simplicial rational polyhedral fan (henceforth, simply fan), with $\Sigma(i)$ denoting the set of i-dimensional faces of Σ, and let X_Σ be the associated toric variety. A Weil divisor on X_Σ is of the form

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho, \text{ with } a_\rho \in \mathbb{Z}.$$

Let $d = |\Sigma(1)|$. The class group of X_Σ has a presentation

$$0 \longrightarrow M \xrightarrow{\psi} \mathbb{Z}^d \xrightarrow{\phi} \mathrm{Cl}(X_\Sigma) \longrightarrow 0,$$

where ψ is defined by

$$\chi^m \mapsto \sum_{\rho \in \Sigma(1)} \langle m, v_\rho \rangle D_\rho, \text{ where } v_\rho \text{ is a minimal lattice generator for } \rho.$$

In [3], Cox introduced the total coordinate ring (henceforth called the *Cox ring*) of X_Σ. This is a polynomial ring, graded by the class group $\mathrm{Cl}(X_\Sigma)$.

Definition 1.1.

$$S = \mathbb{K}[x_\rho \mid \rho \in \Sigma(1)] = \bigoplus_{\alpha \in \mathrm{Cl}(X_\Sigma)} S_\alpha.$$

The utility of this grading is that for $\alpha \simeq D \in \mathrm{Cl}(X_\Sigma)$, $H^0(\mathcal{O}_X(D)) \simeq S_\alpha$. For more background on toric varieties, see [4], [5], or [7].

2. Cohomology and the Cox ring

The Cox ring has a distinguished ideal, the *toric irrelevant ideal*

$$B(\Sigma) = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma \rangle, \text{ where } x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho.$$

Note that $B(\Sigma)$ is generated by monomials corresponding to the complements of the maximal faces of Σ. For an ideal $I = \langle f_1, \ldots, f_m \rangle$ let

$$I^{[l]} = \langle f_1^l, \ldots, f_m^l \rangle.$$

In [6], Eisenbud-Mustaţă-Stillman show that for $D \in \mathrm{Cl}(X_\Sigma)$, $i \geq 1$ and $l \gg 0$,

$$H^i(\mathcal{O}_X(D)) \simeq Ext_S^{i+1}(S/B(\Sigma)^{[l]}, S(D))_0, \tag{2}$$

They also obtain a bound for l. Fix a basis for M, and let A be a $d \times n$ matrix with a row for each ray $u_\rho \in \Sigma(1)$, written with respect to the fixed basis. Define

$$\begin{aligned} a &= \max(|\text{entries of } A|) \\ b &= \max(|(n-1) \times (n-1) \text{ minors of } A|) \\ c &= \min(|\text{nonzero } n \times n \text{ minors of } A|). \end{aligned} \tag{3}$$

Corollary 3.3 of [6] shows that if $D = \sum_\rho a_\rho D_\rho$, then Equation 2 holds for

$$l \geq n^2 \max_{\rho \in \Sigma(1)} (|a_\rho|) ab/c \tag{4}$$

For brevity, we use lower case to denote $\dim_{\mathbb{K}}$ of an object, e.g. $s_\alpha = \dim_{\mathbb{K}} S_\alpha$.

Lemma 2.1. *For $l \gg 0$ and $D \in \mathrm{Cl}(X_\Sigma)$,*

$$\chi(\mathcal{O}_X(D)) = \sum_{i=0}^{n} (-1)^i h^i(D) = s_D - \sum_{i=0}^{n+1} (-1)^i ext_S^i(S/B(\Sigma)^{[l]}, S(D))_0. \tag{5}$$

Proof. $Ext_S^0(S/B(\Sigma)^{[l]}, S) = Ext_S^1(S/B(\Sigma)^{[l]}, S) = 0$, so this follows from [6]. □

Lemma 2.2. *If $F_\bullet$ is a free resolution for $S/B(\Sigma)^{[l]}$, then*

$$\begin{aligned} \sum_{i=0}^{n+1} (-1)^i ext_S^i(S/B(\Sigma)^{[l]}, S(D))_0 &= \sum_{i=0}^{d} (-1)^i \dim_{\mathbb{K}} F_i^\vee(D)_0 \\ &= \sum_{i=0}^{d} (-1)^i \dim_{\mathbb{K}} (F_i)_D^\vee. \end{aligned} \tag{6}$$

Proof. Take Euler characteristics. □

Lemma 2.3. *If $F_\bullet$ is a minimal free resolution for $S/B(\Sigma)^{[l]}$, then*

$$\dim_{\mathbb{K}} (F_i)_D^\vee = \sum_{D' \in \mathrm{Cl}(X_\Sigma)} tor_i^S(S/B(\Sigma)^{[l]}, \mathbb{K})_{D'} \cdot s_{D'+D}.$$

Proof. Let $F_\bullet$ be a minimal free resolution for $S/B(\Sigma)^{[l]}$, and

$$r_i(D') = tor_i^S(S/B(\Sigma)^{[l]}, \mathbb{K})_{D'}.$$

Then

$$F_i = \bigoplus_{D' \in \mathrm{Cl}(X_\Sigma)} S(-D')^{r_i(D')}.$$

Now dualize and take the shift by D into account. □

3. Combinatorial commutative algebra

Taylor resolution. We now observe that the multigraded betti numbers $r_i(D')$ of $S/B(\Sigma)^{[l]}$ can be replaced with related numbers which arise from a Taylor resolution for $S/B(\Sigma)$. The Taylor resolution [23] of a monomial ideal is a variant of the Koszul complex, which takes into account the LCM's of the monomials involved.

Let $I = \langle m_1, \ldots, m_k \rangle$ be a monomial ideal, and consider a complete simplex with vertices labelled by the m_i, and each n-face F labelled with the LCM of the $n+1$ monomials corresponding to vertices of F. Define a chain complex where the differential on an n-face $F = [v_{i_0}, \ldots, v_{i_n}]$ is

$$d(F) = \sum_{j=0}^{n} (-1)^j \frac{m_F}{m_{F \setminus v_{i_j}}} F \setminus v_{i_j},$$

with m_F denoting the monomial labelling face F. As shown by Taylor, this complex is actually a resolution (though often nonminimal) of I. When the m_i are squarefree, the LCM of a subset of l^{th} powers is the l^{th} power of the LCM of the original monomials, hence the Taylor resolution for $I^{[l]}$ is given by the l^{th} power of the Taylor resolution for I, in the sense that a summand $S(-\alpha)$ in the free resolution for I is replaced with $S(-l \cdot \alpha)$ in the resolution for $I^{[l]}$.

Thus, the Taylor resolution of $S/B(\Sigma)$ determines the Taylor resolution of $S/B(\Sigma)^{[l]}$. The formula in Lemma 2.3 requires a minimal free resolution, which the Taylor resolution is generally not. However, this is no obstacle:

Lemma 3.1. *If $F_\bullet$ is a free resolution for $S/B(\Sigma)$, then*

$$\sum_{i=0}^{n+1} (-1)^i ext^i_S(S/B(\Sigma)^{[l]}, S(D))_0 = \sum_{i=0}^{d} (-1)^i \sum_{D' \in \mathrm{Cl}(X_\Sigma)} tor_i^S(S/B(\Sigma), \mathbb{K})_{D'} \cdot s_{l \cdot D' + D}.$$

Proof. If $F_\bullet$ is a minimal resolution of $S/B(\Sigma)^{[l]}$, then Lemmas 2.2 and 2.3 yield

$$\sum_{i=0}^{n+1} (-1)^i ext^i_S(S/B(\Sigma)^{[l]}, S(D))_0 = \sum_{i=0}^{d} (-1)^i \sum_{D' \in \mathrm{Cl}(X_\Sigma)} tor_i^S(S/B(\Sigma)^{[l]}, \mathbb{K})_{D'} \cdot s_{D' + D}.$$

Lemma 2.2 shows that the l^{th} power of a Taylor resolution for $S/B(\Sigma)$ can be used to compute the left-hand side. Furthermore, when $F_\bullet$ is non-minimal, in the expression

$$\sum_{i=0}^{d} (-1)^i \dim_{\mathbb{K}} (F_i)^\vee_D$$

the nonminimal summands cancel out, hence we may pass back to the description in terms of Tor, yielding the result. $\square$

Alexander duality and monomial ideals. Let Δ be a simplicial complex on vertex set $\{1, \ldots, d\}$. Let $S = \mathbb{Z}[x_1, \ldots, x_d]$ be a polynomial ring, with variables corresponding to the vertices of Δ.

Definition 3.2. *The Stanley-Reisner ideal $I_\Delta \subseteq S$ is the ideal generated by all monomials corresponding to nonfaces of Δ:*

$$I_\Delta = \langle x_{i_1} \cdots x_{i_k} | [i_1, \ldots, i_k] \textit{ is not a face of } \Delta \rangle.$$

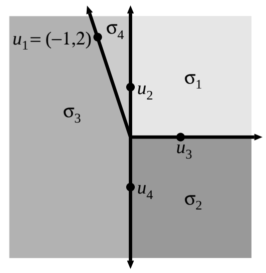

FIGURE 1. The fan for $\mathcal{H}_2$

The Stanley-Reisner ring is S/I_Δ. The intersection of a complete simplicial fan $\Sigma \subseteq \mathbb{R}^n$ with the unit sphere S^{n-1} gives a simplicial complex we denote by P_Σ; define I_Σ as the Stanley-Reisner ideal of P_Σ.

Definition 3.3. *If Δ is a simplicial complex on $[d] = \{1, \ldots d\}$, then the Alexander dual $\Delta^\vee$ is a simplicial complex consisting of the complements of the nonfaces of Δ:*

$$\Delta^\vee = \{[d] \setminus \sigma | \sigma \notin \Delta\}.$$

Example 3.4. The Hirzebruch surface $\mathcal{H}_2$ corresponds to the fan in Figure 1. Since $[u_2, u_4]$ and $[u_1, u_3]$ are nonfaces of Σ, and every other nonface such as $[u_1, u_2, u_4]$ contains them, the Stanley-Reisner ideal is

$$I_\Sigma = \langle x_1x_3, x_2x_4 \rangle.$$

The Alexander dual $\Sigma^\vee$ contains all $\rho \in \Sigma(1)$. Since $\widehat{u_1u_3} = [u_2, u_4]$ and $\widehat{u_2u_4} = [u_1, u_3]$, $\Sigma^\vee(2) = \{[u_2, u_4], [u_1, u_3]\}$. So

$$I_{\Sigma^\vee} = \langle x_1x_2, x_1x_4, x_2x_3, x_3x_4 \rangle.$$

Lemma 3.5. *The toric irrelevant ideal $B(\Sigma)$ is Alexander dual to the Stanley-Reisner ideal I_Σ.*

Proof. The Alexander dual $I_{\Sigma^\vee}$ to I_Σ is obtained by monomializing ([15], Proposition 1.35) a primary decomposition for I_Σ. If $MC(\Sigma)$ denotes the set of minimal cofaces of Σ, then the primary decomposition of I_Σ is

$$I_\Sigma = \bigcap_{[i_1,\ldots,i_k] \in MC(\Sigma)} \langle x_{i_1}, \ldots, x_{i_k} \rangle.$$

The ideal $I_{\Sigma^\vee}$ is generated by monomials corresponding to minimal cofaces, which are complements to maximal faces, hence $I_{\Sigma^\vee} = B(\Sigma)$. □

Theorem 3.6 (Danilov [5], Jurkiewicz [10])**.** *For a complete simplicial fan Σ, let $J = \langle div(\chi^{\mathbf{m}}) | \mathbf{m} \in M \rangle$. The rational Chow ring $Ch(X_\Sigma)$ is the rational Stanley-Reisner ring of Σ, modulo J.*

The ideal J is minimally generated by a regular sequence; it is these linear forms which encode the geometry of Σ. To interpret the Euler characteristic of $\mathcal{O}_X(D)$ in terms of intersection theory, we must change computations involving the toric irrelevant ideal into computations involving the Stanley-Reisner ideal. For a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_d]$ endowed with the fine (also called $\mathbb{Z}^d$) grading

$\deg(x_i) = \mathbf{e}_i \in \mathbb{Z}^d$ and squarefree monomial ideal M, the following result of Mustaţă [[16], Corollary 3.1] provides the bridge:

$$Tor_i^R(M^\vee, \mathbb{K})_{\mathbf{m}} \simeq Ext_R^{|\mathbf{m}|-i}(R/M, R)_{-\mathbf{m}} \text{ if } \mathbf{m} \in \{0,1\}^d, \text{ else } 0. \tag{7}$$

Letting $Z = \{0,1\}^d$, applying Mustaţă's result yields:

$$\begin{aligned} tor_i^S(S/B(\Sigma), \mathbb{K})_{D'} &= \sum_{\substack{\mathbf{m}\in Z,\\ \phi(\mathbf{m})=D'}} tor_i^S(S/B(\Sigma), \mathbb{K})_{\mathbf{m}} \\ &= \sum_{\substack{\mathbf{m}\in Z,\\ \phi(\mathbf{m})=D'}} ext_S^{|\mathbf{m}|-i+1}(S/I_\Sigma, S)_{-\mathbf{m}} \end{aligned} \tag{8}$$

Lemma 3.7. *For a complete fan $\Sigma \subseteq N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ with $|\Sigma(1)| = d$,*

1. *$Ext_S^j(S/I_\Sigma, S) = 0$ for all $j \neq d-n$.*
2. *In the $\mathbb{Z}^d$ grading, $Ext_S^{d-n}(S/I_\Sigma, S) \simeq S/I_\Sigma(\mathbf{1})$.*

Proof. From Definition 3.2, I_Σ is the Stanley-Reisner ideal of the simplicial sphere P_Σ, which is Gorenstein by Corollary II.5.2 of [22]. Since $\dim P_\Sigma = n-1$,

$$\operatorname{codim}(I_\Sigma) = (d-1) - (n-1) = d-n.$$

Everything follows from this, save that S/I_Σ is shifted by $\mathbf{1}$. The Gorenstein property means the minimal free resolution of S/I_Σ is of the form

$$0 \longrightarrow S(-\alpha) \xrightarrow{\partial_{d-n}} \bigoplus_{j=1}^{k} S(-\beta_j) \xrightarrow{\partial_{d-n-1}} \cdots \longrightarrow \bigoplus_{j=1}^{k} S(-\gamma_j) \xrightarrow{[I_\Sigma]} S \longrightarrow S/I_\Sigma \longrightarrow 0,$$

where ∂_{d-n} is (up to signs) the transpose of the matrix of minimal generators $[I_\Sigma]$. To show that the shift in Ext^{d-n} is $\mathbf{1}$, we use a result of Hochster. For a complex Δ and weight α, let $\Delta|_\alpha = \{\sigma \in \Delta \mid \sigma \subseteq \alpha\}$. Equating the multidegree $\mathbf{1}$ with the full simplex on all vertices of Δ, Hochster's formula (5.12 of [15]) yields

$$Tor_{d-n}^S(S/I_\Sigma, \mathbb{K})_{\mathbf{1}} = \widetilde{H}^{n-1}(\Sigma|_{\mathbf{1}}, \mathbb{K}).$$

Since $\Sigma|_{\mathbf{1}} \simeq P_\Sigma \simeq S^{n-1}$, the result follows. $\square$

Example 3.8. The Stanley-Reisner ring for the fan Σ of Example 3.4 has a $\mathbb{Z}^4$ graded minimal free resolution

$$0 \longrightarrow S(-1,-1,-1,-1) \xrightarrow{\begin{bmatrix} -x_2x_4 \\ x_1x_3 \end{bmatrix}} \begin{matrix} S(-1,0,-1,0) \\ \oplus \\ S(0,-1,0,-1) \end{matrix} \xrightarrow{\begin{bmatrix} x_1x_3 & x_2x_4 \end{bmatrix}} S \longrightarrow S/I_\Sigma.$$

Thus, $Ext^2(S/I_\Sigma, S) \simeq S(1,1,1,1)/I_\Sigma$. The simplicial complex P_Σ consists of vertices $[1],[2],[3],[4]$ and edges $[12],[23],[34],[41]$ and is homotopic to S^1. Since the multidegrees are all smaller than $\mathbf{1}$ in the pointwise order, $\Sigma|_{\mathbf{1}} = P_\Sigma$, so

$$\mathbb{K} = \widetilde{H}^1(S^1, \mathbb{K}) = \widetilde{H}^1(\Sigma|_{\mathbf{1}}, \mathbb{K}) = Tor_2^S(S/I_\Sigma, \mathbb{K})_{\mathbf{1}},$$

showing the shift α in the last step of the free resolution of S/I_Σ is $S(-\mathbf{1})$.

4. Proof of Equation 1

We now prove Equation 1. By Equation 5,

$$\chi(\mathcal{O}_X(D)) = s_D - \sum_{i=0}^{n+1} (-1)^i ext_S^i(S/B(\Sigma)^{[l]}, S(D))_0.$$

Let $\gamma(\mathbf{m}) = s_{l\cdot\phi(\mathbf{m})+D}$ and $E = \sum_{i=0}^{n+1}(-1)^i ext_S^i(S/B(\Sigma)^{[l]}, S(D))_0$. It suffices to show

$$E = s_D + \sum_{\mathbf{m}\in Z\setminus\mathbf{0}} (-1)^{|\mathbf{m}|-d+n+1} \dim_{\mathbb{K}}(S/I_\Sigma)_{\mathbf{1-m}} \cdot \gamma(\mathbf{m}).$$

First, observe that

$$\begin{aligned} E &= \sum_{i=0}^{d}(-1)^i \sum_{D'\in \mathrm{Cl}(X_\Sigma)} \Bigg(\sum_{\substack{\mathbf{m}\in Z,\\ \phi(\mathbf{m})=D'}} tor_i^S(S/B(\Sigma),\mathbb{K})_{\mathbf{m}} \Bigg) \cdot \gamma(\mathbf{m}). \\ &= \sum_{i=0}^{d}(-1)^i \sum_{\mathbf{m}\in Z} tor_i^S(S/B(\Sigma),\mathbb{K})_{\mathbf{m}} \cdot \gamma(\mathbf{m}). \\ &= s_D + \sum_{i=1}^{d}(-1)^i \sum_{\mathbf{m}\in Z\setminus\mathbf{0}} tor_i^S(S/B(\Sigma),\mathbb{K})_{\mathbf{m}} \cdot \gamma(\mathbf{m}). \end{aligned} \tag{9}$$

The first line follows from Lemma 3.1, the second line is simply a rearrangement, and the third line follows from the observation that

$$s_D = tor_0^S(S/B(\Sigma),\mathbb{K})_{\mathbf{0}} \cdot \gamma(\mathbf{0}).$$

For $i \geq 0$,

$$Tor_i^S(B(\Sigma),\mathbb{K}) \simeq Tor_{i+1}^S(S/B(\Sigma),\mathbb{K}),$$

so using Equation 7 we may rewrite the last line of Equation 9 as

$$s_D + \sum_{i=0}^{d-1}(-1)^{i+1} \sum_{\mathbf{m}\in Z\setminus\mathbf{0}} ext_S^{|\mathbf{m}|-i}(S/I_\Sigma, S)_{-\mathbf{m}} \cdot \gamma(\mathbf{m}). \tag{10}$$

By Lemma 3.7, $Ext_S^{|\mathbf{m}|-i}(S/I_\Sigma, S)$ is nonzero iff $|\mathbf{m}| - i = d - n$, and

$$Ext_S^{d-n}(S/I_\Sigma, S) \simeq S/I_\Sigma(\mathbf{1}).$$

Since the only nonzero terms in Equation 10 occur for $i = |\mathbf{m}| - d + n$ we rewrite Equation 10 as

$$\begin{aligned} &= s_D + \sum_{\mathbf{m}\in Z\setminus\mathbf{0}} (-1)^{|\mathbf{m}|-d+n+1} ext_S^{d-n}(S/I_\Sigma, S)_{-\mathbf{m}} \cdot \gamma(\mathbf{m}) \\ &= s_D + \sum_{\mathbf{m}\in Z\setminus\mathbf{0}} (-1)^{|\mathbf{m}|-d+n+1} \dim_{\mathbb{K}}(S/I_\Sigma)_{\mathbf{1-m}} \cdot \gamma(\mathbf{m}) \end{aligned} \tag{11}$$

This shows that

$$E = s_D + \sum_{\mathbf{m}\in Z\setminus\mathbf{0}} (-1)^{|\mathbf{m}|-d+n+1} \dim_{\mathbb{K}}(S/I_\Sigma)_{\mathbf{1-m}} \cdot \gamma(\mathbf{m}),$$

and Equation 1 follows. □

Example 4.1. Consider the divisor $D = 3D_3 - 5D_4$ on the Hirzebruch surface $\mathcal{H}_2$ from Figure 1. Since the support function for D is not convex, D is not nef. Thus, computing $\chi(\mathcal{O}_{\mathcal{H}_2}(D))$ involves more than a simple global section computation. A direct calculation with Riemann-Roch for surfaces shows that

$$\chi(\mathcal{O}_{\mathcal{H}_2}(D)) = 4.$$

Using the methods of §9.4 of [4], it can be shown that $h^0(D) = 0$, $h^1(D) = 2$, and $h^2(D) = 6$. Now we illustrate how to apply Equation 1. Let

$$\phi = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

so that the Class group is given by

$$\mathbb{Z}^4 \xrightarrow{\phi} \mathbb{Z}^2 \simeq Cl(\mathcal{H}_2) \longrightarrow 0.$$

The Eisenbud-Mustaţă-Stillman bound of Equation 4 is $l = 80$, but a careful analysis (see Example 3.6 of [6]) shows that in this case taking $l = 4$ is sufficient. Then for example with $\mathbf{m} = (0,1,0,1)$ we have $\phi(\mathbf{m}) = (-2,2)$ so since $D = (3,-5)$,

$$S_{4\cdot\phi(\mathbf{m})+D} = S_{(-5,3)} = H^0(\mathcal{O}_{\mathcal{H}_2}(-5,3)),$$

and the dimension of this space is two. However,

$$(S/I_\Sigma)_{\mathbf{1}-(0,1,0,1)} = (S/I_\Sigma)_{(1,0,1,0)} = 0,$$

since $x_1x_3 \in I_\Sigma$. A check shows that all terms in the summation vanish, save when

$$\mathbf{m} \in \{(1,1,0,1),(0,1,1,1),(1,1,1,1)\}$$

For the first two values, $\phi(\mathbf{m}) = (-1,2)$, and we compute

$$S_{4\cdot\phi(\mathbf{m})+D} = S_{(-1,3)} = H^0(\mathcal{O}_{\mathcal{H}_2}(-1,3)),$$

which has dimension twelve. Since $\mathbf{1} - \mathbf{m}$ is either $(0,0,1,0)$ or $(1,0,0,0)$, for these two values of $\mathbf{m}$,

$$\dim_{\mathbb{K}}(S/I_\Sigma)_{\mathbf{1}-\mathbf{m}} = 1$$

Since $|\mathbf{m}| - d + n = 1$, these two weights contribute $(-1)\cdot 2\cdot 12 = -24$ to the Euler characteristic. For the remaining weight $\mathbf{m} = (1,1,1,1)$, the Stanley-Reisner ring is one dimensional in degree $\mathbf{1} - \mathbf{m} = (0,0,0,0)$, and $\phi(1,1,1,1) = (0,2)$ and

$$S_{4\cdot\phi(\mathbf{m})+D} = S_{(3,3)} = H^0(\mathcal{O}_{\mathcal{H}_2}(3,3)),$$

which has dimension 28. Since $|\mathbf{m}| - d + n = 2$ the contribution is positive, thus

$$\chi(\mathcal{O}_{\mathcal{H}_2}(3D_3 - 5D_4)) = -24 + 28 = 4.$$

Problem As noted in the introduction, this work began as an attempt to find a toric proof of Hirzebruch-Riemann-Roch using Equation 1; it would be interesting to find such a proof. A proof of Equation 1 also follows from results of Maclagan-Smith [14], I thank Greg Smith for noting this.

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